Matrices: Geometric Interpretation

Start with a vector of length 2, for example, \( x = (1, 2) \). This among other things give
the coordinates for a point on a plane. Take a \( 2 \times 2 \) matrix, for example,
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]
Then \( Ax = (1, -2) \). That is, multiplication by the matrix has created a new set of
coordinates (More generally, if \( A \) is an \( n \times n \) square matrix, and \( x \) is vector of length
\( n \), \( Ax \) is also a vector of length \( n \)). A matrix can be thought of as an operator moving
points around the plane. The matrix \( A \) above represents a reflection in the \( x \)-axis.
Other matrices will represent other geometric operations such as stretches, rotations
and so on.

\( \text{Eigenvalues} \) (aka characteristic values) and \( \text{eigenvectors} \) (aka characteristic vectors)
enable these operations to be characterised relatively compactly. An eigenvector of a
matrix is a vector which is left unchanged in direction (but not necessarily in magni-
tude) under the transformation defined by that matrix. For example, the vector \((1,0)\)
is an eigenvector for \( A \) above, because \( A(1,0) = (1,0) \). \( A \) represents a reflection in the
\( x \)-axis, and so points on the \( x \)-axis like \((1,0)\) do not move. If
\[
B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.
\]
then \( B(1,0) = (2,0) \). \( B \) again represents a reflection in the \( x \)-axis but with some
stretching. So \((1,0)\) is also an eigenvector for \( B \), it gets stretched to \((2,0)\) but all that
has happened to it is to be multiplied by 2. This factor by which the eigenvector is
multiplied is the eigenvalue associated with that eigenvector.

Calculating Eigenvalues

Formally, an eigenvector for a square matrix \( A \) is a non-zero vector that satisfies the
equation
\[
Ax = \lambda x
\]
where \( \lambda \) is a scalar and is the associated eigenvalue. Every \( n \times n \) matrix has \( n \) eigenvalues
and \( n \) (sets of) eigenvectors. This latter fact is because eigenvectors are not unique. If
\( x \) is an eigenvector corresponding to an eigenvalue \( \lambda \), then, for example, \( 3x, x/2, -2x \)
are all eigenvectors for that eigenvalue as well. For example, for \( A \) above, all vectors
of the form \((a,0)\) are unchanged under the operation defined by \( A \).
Starting with the eigenvalue equation (1), subtract $\lambda x$ from both sides to obtain

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0.$$  \hspace{1cm} (2)

($I$ is the identity matrix). Matrix theory says that if for some matrix $B$ and for some non-zero vector $x$, $Bx = 0$, then $B$ is a singular matrix. That is, its determinant is zero. Therefore the matrix

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$  \hspace{1cm} (3)

has a zero determinant. That is,

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$  \hspace{1cm} (4)

which is a quadratic equation in $\lambda$. This equation has two solutions, which will be the eigenvalues of the matrix. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$  \hspace{1cm} (5)

then we have $(1 - \lambda)(-1 - \lambda) - 4 = 0$ or $\lambda^2 - 5 = 0$ which has solutions $\lambda = \pm \sqrt{5}$.

**Useful Facts**

Note that if the matrix is diagonal ($a_{12} = a_{21} = 0$) or triangular (either $a_{12}$ or $a_{21}$ is zero), then the above reduces to $(a_{11} - \lambda)(a_{22} - \lambda) = 0$. This equation has two clear solutions $\lambda = a_{11}$ and $\lambda = a_{22}$. That is, the eigenvalues are the diagonal elements. The same principal applies to all $n \times n$ matrices.

1. If a matrix is **DIAGONAL**
2. Or if a matrix is **TRIANGULAR**

then the eigenvalues are just the diagonal elements.

Two other important facts

1. The sum of the eigenvalues of a matrix is equal to the sum of its diagonal elements, which is called the **trace** of a matrix.
2. The product of the eigenvalues of a matrix is equal to the determinant of the matrix.
In the example (5) above, the trace is \(0 = \sqrt{5} + (-\sqrt{5})\) and the determinant is \(-5 = \sqrt{5} \times -\sqrt{5}\).

The second point implies: IF A MATRIX HAS EVEN ONE ZERO EIGENVALUE, IT IS SINGULAR!

**Diagonalisation and Normal Forms**

Suppose there are \(n \times n\) matrices \(A\) and \(P\), then note that \(A\) and (if \(P^{-1}\) exists) \(P^{-1}AP\) have the same eigenvalues. One says that a matrix \(A\) is diagonalisable if there exists a matrix \(P\) such that

\[
P^{-1}AP = D
\]

where \(D\) is a diagonal matrix. Note that the eigenvalues of \(D\) are just its diagonal elements. But as \(A\) and \(D\) have the same eigenvalues, if \(A\) is diagonalisable, then \(D\) has \(A\)'s eigenvalues along its diagonal.

**Symmetric Matrices:**

We have the following result.

**Theorem 1** An \(n \times n\) symmetric matrix \(A\)

1. has \(n\) real eigenvalues \(\lambda_1, \ldots, \lambda_n\)
2. there are \(n\) orthogonal eigenvectors (i.e. \(x_i \cdot x_j = 0\) for \(j \neq i\)).
3. There exists a matrix \(P\), for which the columns are eigenvectors of \(A\), such that \(D = P^{-1}AP\).

Example: the matrix

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}
\]

has eigenvalues \((3,-1)\) and corresponding eigenvectors \((1,1)\) and \((-1,1)\) respectively. Thus,

\[
P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}/2, \quad P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}
\]
Non-Symmetric Matrices:

We have the following result.

**Theorem 2 (Jordan)*** For any \( n \times n \) matrix \( A \), there exists a matrix \( P \), such that \( J = P^{-1}AP \) (\( J \) is the “Jordan normal form”), where \( J = D + N \) where \( D \) is a diagonal matrix with the eigenvalues of \( A \) and \( N \) is nilpotent (i.e. \( N^k = 0 \) for some positive integer \( k \)).

Example: the matrix

\[
A = \begin{bmatrix}
4 & 1 \\
-1 & 6
\end{bmatrix}
\]

has repeated eigenvalues (5,5) and only one independent eigenvector (1,1). Nonetheless, we have

\[
P = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix},
\]

\[
P^{-1} = \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix},
\]

\[
P^{-1}AP = \begin{bmatrix}
5 & 1 \\
0 & 5
\end{bmatrix} = \begin{bmatrix}
5 & 0 \\
0 & 5
\end{bmatrix} + \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

NB here our matrix \( N \) is such that \( N^2 = 0 \).

**Complex Eigenvalues**

Asymmetric matrices may have complex eigenvalues. That is, they may have eigenvalues of the form \( \alpha + \beta i \), where \( \alpha, \beta \) are normal numbers but \( i \) is the imaginary number defined as \( \sqrt{-1} \). For example, the matrix

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

represents geometrically a rotation of 90\(^\circ\) anticlockwise. It has a zero trace but its determinant is 1. So \( \lambda_1 + \lambda_2 = 0 \), and \( \lambda_1\lambda_2 = 1 \). Combining these two equations, you can obtain \( \lambda_1^2 = -1 \) or the two eigenvalues are equal to \( \pm \sqrt{-1} = \pm i \), where \( i \) represents the square root of \( -1 \).

This illustrates several points about complex eigenvalues

1. Complex eigenvalues are associated with circular and cyclical motion.
2. Complex eigenvalues come in pairs. If \( \alpha + \beta i \) is an eigenvalue for a matrix, then \( \alpha - \beta i \) is also.
3. Note that the sum of such a pair (i.e \( \alpha + \beta i + \alpha - \beta i = 2\alpha \)) and the product of such a pair (i.e. \( (\alpha + \beta i)(\alpha - \beta i) = \alpha^2 + \beta^2 \)) are both real numbers and are not complex. It is still true that the sum of complex eigenvalues equals the trace of the matrix and that the product of the eigenvalues equals the determinant.
4. Eigenvectors associated with complex eigenvalues are themselves complex.

5. The Jordan Theorem (Theorem 2) still applies.

Example: the matrix
\[ A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \]
has eigenvalues \((5 \pm \sqrt{3}i)/2\) and corresponding eigenvectors \((-3 + (5 + \sqrt{3}i)/2, 1)\) and \((-3 + (5 - \sqrt{3}i)/2, 1)\). These are not orthogonal but
\[
P = \begin{bmatrix} \frac{-3 + (5 + \sqrt{3}i)/2}{1} & -3 + (5 - \sqrt{3}i)/2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -i/\sqrt{3} & 1/2 - i\sqrt{3}/6 \\ i/\sqrt{3} & 1/2 + i\sqrt{3}/6 \end{bmatrix}
\]
and
\[
P^{-1}AP = \begin{bmatrix} \frac{-3 + (5 + \sqrt{3}i)/2}{1} & 0 \\ 0 & -3 + (5 - \sqrt{3}i)/2 \end{bmatrix}
\]
so \(A\) is diagonalisable.

**Application: Eigenvalues and Quadratic Forms**

Remember that a matrix is called *positive definite* if
\[
x \cdot Ax > 0
\]
for all non-zero vectors \(x\). There is a link between this and the eigenvalues of a matrix. Take the eigenvalue equation (1) and multiply both sides by \(x\) to obtain
\[
x \cdot Ax = \lambda x \cdot x.
\]
Now if \(A\) is positive definite, the lefthand side of the equation is positive. Now, \(x \cdot x = \sum x_i^2 > 0\), so \(\lambda\) must be positive. We have just proved,

if a matrix is positive (negative) definite, all its eigenvalues are positive (negative).

If a matrix is symmetric, we can add the following

If a symmetric matrix has all its eigenvalues positive (negative), it is positive (negative) definite.

**Proof:** If a matrix is symmetric it has \(n\) orthogonal eigenvectors \(z_1, \ldots, z_n\). Then the eigenvectors form a *basis* for \(\mathbb{R}^n\). That is, any vector \(x \in \mathbb{R}^n\) can be written \(x = \sum_{i=1}^{n} a_i z_i\) for suitable constants \(a_1, \ldots, a_n\). We thus have
\[
x \cdot Ax = \sum_{i=1}^{n} a_i z_i \cdot A \sum_{i=1}^{n} a_i z_i = \sum_{i=1}^{n} a_i^2 \lambda_i z_i \cdot z_i
\]
which is strictly positive (negative) if all eigenvalues are positive (negative).

If a matrix is not symmetric, this does not apply. For example, the matrix

\[ A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \]

is triangular so that its eigenvalues are 1 and 2, but if \( x = (1, 1) \) we have \( x \cdot Ax = -98 < 0 \) so that \( A \) is not positive definite.

**Application: Difference Equations**

Let us look at linear difference equations with \( n \) variables. Time is discrete \( t = 0, 1, 2, \ldots \) and we write the matrix difference equation as

\[ x_{t+1} = Ax_t = A^t x_0 \]  \hspace{1cm} (10)

for some \( n \times n \) matrix \( A \). First suppose that \( A \) is diagonal with diagonal elements \( a_1, \ldots, a_n \) then

\[ x_{it} = a_i^t x_0. \]  \hspace{1cm} (11)

That is, if the matrix \( A \) is diagonal we have an exact solution to the difference equation that is easily calculable.

Suppose a matrix \( A \) is not diagonal but is diagonalisable, so that \( P^{-1}AP = D \). Then define \( y_t = P^{-1}x_t \) for any \( t \) so that \( x_t = Py_t \). Then (10) becomes

\[ Py_{t+1} = APy_t \]  \hspace{1cm} (12)

or

\[ y_{t+1} = P^{-1}APy_t = Dy_t \]  \hspace{1cm} (13)

That is, we have constructed a diagonal dynamic system. From (11), we know that has the solution \( y_{it} = \lambda_i^t y_{i0} \) where \( \lambda_i \) is the \( i \)th eigenvalue of \( A \) (remember \( A \) and \( D \) have the same eigenvalues). But we convert this back and get a solution for \( x_t \)

\[ x_t = Py_t = P \begin{bmatrix} \lambda_1^t y_{10} \\ \vdots \\ \lambda_n^t y_{n0} \end{bmatrix} \]  \hspace{1cm} (14)

or \( x_{it} = \sum_{i=1}^{n} k_i \lambda_i^t \) where the \( k_i \) are constants determined by \( P \) and by initial values of \( x \).

Suppose a matrix \( A \) is not diagonal nor diagonalisable, but by the Jordan Theorem we have \( P^{-1}AP = D + N \). Then define \( y_t = P^{-1}x_t \) for any \( t \) so that \( x_t = Py_t \). Then (10) becomes

\[ y_{t+1} = P^{-1}APy_t = (D + N)y_t \]  \hspace{1cm} (15)
That is, we have constructed an almost diagonal dynamic system. How does this work? Well,

\[ y_{t+1} = (D + N)^t y_0 = (D^t + tD^{t-1}N + ... + N^t)y_0 \]

Now, remember \( N \) is nilpotent so that in the above equation the higher powers of \( N \) are zero. For example (and this is typical), \( N^t = 0 \) for \( t \geq 2 \). Then, it would be the case that

\[ x_{it} = \sum_{i=1}^{n} k_i \lambda_i^t + t \sum_{i=1}^{n} m_i \lambda_i^{t-1} \]  

(16)

where the \( k_i \) and \( m_i \) are constants.

So, under the the Jordan Theorem, any solution to the general difference equation (10) can be written something like (16), that is in powers of the eigenvalues of \( A \). If \( |\lambda_i| < 1 \), then \( \lim_{t \to \infty} \lambda_i^t = 0 \), if \( |\lambda_i| > 1 \), then \( \lim_{t \to \infty} \lambda_i^t = \infty \). Thus, we have the following

**Theorem 3** Assume \( A \) is non-singular. Then, the linear difference equation (10) has a unique fixed point \( x = 0 \). If the eigenvalues of \( A \) satisfy \( |\lambda_i| < 1 \) for \( i = 1, ..., n \) then all solutions of (10) from any initial conditions converge to 0.

**Application: Differential Equations**

Consider a linear differential equation of the form

\[ \dot{x} = Ax \]  

(17)

where \( x \in \mathbb{R}^n \) and \( A \) is a \( n \times n \) matrix. First, suppose \( A \) is diagonal. Then it is relatively easy to see that the differential equation will have solutions of the form \( x_i(t) = k_i e^{\lambda_i t} \) for \( i = 1, ..., n \), where the \( \lambda_i \) are the diagonal elements, equivalently the eigenvalues of \( A \) and the \( k_i \) are constants to be determined by initial conditions.

Suppose again that the matrix \( A \) is not diagonal but is diagonalisable, so that \( P^{-1}AP = D \). Then define \( y(t) = P^{-1}x(t) \) for any \( t \) so that \( x(t) = Py(t) \). Then (17), because \( dy/dt = dy/dx \cdot dx/dt \)

\[ \dot{y} = P^{-1}Ax = P^{-1}APy = Dy \]  

(18)

That is, we have constructed a diagonal dynamic system. We know that has the solution \( y_i(t) = e^{\lambda_i t}y_i(0) \) where \( \lambda_i \) is the \( i \)th eigenvalue of \( A \) (remember \( A \) and \( D \) have the same eigenvalues). But we convert this back and get a solution for \( x(t) \) so that \( x_i(t) = \sum_{i=1}^{n} k_i e^{\lambda_i t} \) where the \( k_i \) are constants determined by \( P \) and by initial values of \( x \).
Suppose \( n = 2 \) and the eigenvalues are complex, then the solution becomes

\[
x_i(t) = k_1 e^{(\alpha + \beta i)t} + k_2 e^{(\alpha - \beta i)t} = e^{\alpha t}(k_1 e^{\beta t} + k_2 e^{-\beta t})
\]

Euler’s formula is that

\[
e^{(\alpha + \beta i)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)
\]

a special case of which is

\[
e^{i\pi} = -1
\] (19)

which gives you \( e, \pi \) and \( i \) all in the same equation!

So, the solution can be written as

\[
x_i(t) = e^{\alpha t}(k_3 \cos \beta t + k_4 \sin \beta t)
\]

where again \( k_3 \) and \( k_4 \) are constants to be determined by initial conditions.

Points worth knowing

1. The functions \( \sin \) and \( \cos \) give rise to undulating waves.

2. Since the whole solution is multiplied by \( e^{\alpha t} \) the solution explodes if \( \alpha > 0 \) and converges if \( \alpha < 0 \). If \( \alpha = 0 \) the solution has oscillations of constant size and neither converges or diverges. IE it is the real part \( \alpha \) of the complex eigenvalue \( \alpha + \beta i \) that determines stability.

So, whether eigenvalues are real or complex, it is whether the real part is positive or negative that determines whether the solution to the differential equation converges or diverges. This leads to the following general result.

**Theorem 4** Assume \( A \) is non-singular. Then, the linear differential equation (17) has a unique fixed point \( x = 0 \). If all the eigenvalues of \( A \) have real part negative then all solutions of (17) from any initial conditions converge to 0. More generally, if there are no zero eigenvalues, the stable manifold of the fixed point 0 is of the same dimension as the number of negative eigenvalues.

Example: the matrix given in (6) was

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}
\]

and had eigenvalues (3,-1) and corresponding eigenvectors (1,1) and (-1,1) respectively. So, solutions to the differential equation \( \dot{x} = Ax \) will be of the form \( x_i(t) = k_{i1} e^{3t} + k_{i2} e^{-t} \) and in general will diverge from \( (0,0) \). However, if the initial conditions correspond to the eigenvector of the negative eigenvalue, e.g. \((-x_1, x_1)\), solutions will be of the form \( x_i(t) = k_2 e^{-t} \) and will converge to zero. NB if e.g. \( x(0) = (-1, 1) \) then from (7), we have \( y(0) = P^{-1}(-1, 1) = (0, 2) \), that is, no weight is placed on the term in \( e^{3t} \), the unstable eigenvalue.